# Lesson 13. A Very Brief Introduction to Markov Processes

## 1 Overview

- Previous lessons: Markov chains
  - Focus on state changes
  - Time between state changes is assumed to be constant
- This lesson: **Markov processes**
  - Similar to Markov chains, but also model the time between state changes
  - We will use Markov processes as a framework to study **queueing processes**

## 2 Markov processes

- Continuous-time, discrete-state stochastic process
- The **state space** is $\mathcal{M} = \{0, 1, 2, \dots, m\}$
  - By convention, we include 0 and the possibility that $m = +\infty$
  - For example, the state might represent number of customers in a queue
- The **transition rate** from state $i$ to state $j$ ($i \neq j$) is

$$g_{ij} = \text{expected number of } i \to j \text{ transitions per unit time} \qquad \text{(e.g. 3 transitions per day)}$$

- Therefore,

$$\frac{1}{g_{ij}} = \text{expected time between } i \to j \text{ transitions} \qquad \text{(e.g. } \tfrac{1}{3} \text{ day per transition)}$$

- The **transition time** from state $i$ to state $j$ ($i \neq j$) is

$$H_{ij} \sim \text{Exponential}(g_{ij})$$

  - $H_{ij}$'s are independent
- We transition $i \to j$ if the transition time for $i \to j$ is the <u>smallest out of all the transition times from state $i$</u>
  - In other words, $H_{ij} = \min\{H_{ik} : k = 0, \dots, m; k \neq i\}$
- We can draw a **transition rate diagram** for a Markov process
  - Similar to a transition probability diagram for a Markov chain, but...
  - We use transition <u>rates</u> instead of transition probabilities as the arc labels
  - To clearly signal that the arc labels are transition rates, we enclose them in a box

**Example 1.** Simplexville College maintains 2 vans to be used by faculty and students for travel to conferences, field trips, etc. The time between requests is exponentially distributed with a mean of 1 day. The time a van is used is also exponentially distributed with a mean of 2 days. If someone requests a van and one is not available, then the request is denied and other transportation, not provided by the motor pool, must be found.

a. Suppose two vans are in use. At what rate does the system transition to having one van in use?
b. Model this system as a Markov process by (i) specifying the state space and (ii) drawing the transition rate diagram.

## 3 Steady state probabilities

- The **overall transition rate** out of state $i$ is $g_{ii} = \sum_{j \neq i} g_{ij}$
- The **steady state probability** of being in state $j$:

$$\begin{aligned} \pi_j &= \text{probability of finding the process in state } j \text{ after a long period of time} \\ &= \text{long-run fraction of time the process is in state } j \end{aligned}$$

- How do we compute these probabilities?
  - Over the long run, the transition rate into state $j$ is
  - Over the long run, the transition rate out of state $j$ is
  - These quantities should be equal in steady state

- In matrix form:
  - **G** is the **generator matrix** of the Markov process: $\mathbf{G} = \begin{bmatrix} -g_{00} & g_{01} & \cdots & g_{0m} \\ g_{10} & -g_{11} & \cdots & g_{1m} \\ \vdots & & \ddots & \vdots \\ g_{m0} & g_{m1} & \cdots & -g_{mm} \end{bmatrix}$
  - Then the steady state probabilities can be found by solving

**Example 2.** Find the generator matrix of the Markov process described in Example 1.

**Example 3.** Consider Example 1. In the long run, what fraction of time are there 0 vans in use? 1? 2?

**Example 4.** Consider Example 1. In the long run, what is the average number of vans in use?

## 4 Notes

- This was a very brief introduction to Markov processes, but just enough to get started with queueing processes
- We discussed why we can add rates intuitively
  - We can do this mathematically because of the following property:

    If $H_1 \sim \text{Exponential}(\lambda_1)$ and $H_2 \sim \text{Exponential}(\lambda_2)$, then $\min\{H_1, H_2\} \sim \text{Exponential}(\lambda_1 + \lambda_2)$.
- We assume in this lesson that the entire state space $\mathcal{M}$ is **irreducible**
  - The notion of irreducibility is similar to that for Markov chains
  - For more details, see SMAS Chapter 7

## 5 Exercises

**Problem 1.** Each customer service representative at Jungle.com spends his or her time answering e-mails and taking phone calls. Phone calls receive first priority, so a representative must interrupt tending to his or her e-mail whenever the phone rings. The time between phone calls is exponentially distributed with a mean of 5 minutes. The length of each phone call is exponentially distributed with a mean of 2 minutes.

a. Model how a representative switches between his or her two tasks as a Markov process by (i) specifying the state space and defining what the states mean, and (ii) specifying the transition rates, either by drawing the transition rate diagram or defining the generator matrix.

b. What is the long-run fraction of time each customer service representative spends answering e-mail? Taking phone calls?

**Problem 2** (SMAS Exercise 7.5)**.** The Football State University motor pool maintains a fleet of vans to be used by faculty and students for travel to conferences, field trips, etc. Requests to use a van occur at about 8 per week on average, and a van is used for an average of 2 days (but there is quite a bit of variability around both numbers). If someone requests a van and one is not available, then other transportation, not provided by the motor pool, must be found. The motor pool currently has 4 vans, but due to university restructuring, it has been asked to reduce its fleet. In order to argue against the proposal, the director of the Motor Pool would like to predict how many requests for vans will be denied if the fleet is reduced from 4 to 3.

a. Model the 3-van system as a Markov process by (i) specifying the state space and defining what the states mean, and (ii) specifying the transition rates, either by drawing the transition rate diagram or defining the generator matrix. Assume the time between requests and the time in use are exponentially distributed.

b. At what rate are requests denied?

c. What is the average number of vans in use?

**Problem 3** (SMAS Exercise 7.10)**.** Two automated testing machines work together testing circuit boards. Each one is independently subject to failure. The failure rate of an automated testing machine, when it is in use, is 0.01 per hour, with the actual time to failure being exponentially distributed. The time required to repair an automated testing machine is also exponentially distributed with mean 24 hours, and only one machine can be repaired at a time. When one of the automated testing machines has failed, the other handles all of the work, which increases its failure rate to 0.02 per hour.

a. Model this system as a Markov process by (i) specifying the state space and defining what the states mean, and (ii) specifying the transition rates, either by drawing the transition rate diagram or defining the generator matrix.

b. What is the long-run fraction of time that both testing machines are not working?

c. What is the long-run fraction of time that at least one testing machine is not working?